A REMARK ON THE GENERALIZED GELFAND TRANSFORMS ON A FUNCTION SPACE

MINORU MATSUBARA*

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1. Introduction.

Let X be a compact Hausdorff space, and $C(X)$ the space of all continuous real-valued functions on X. For f in $C(X)$, we define its norm by $\|f\|_\infty = \sup\{|f(x)|: x \in X\}$ (sup-norm).

A subset E of $C(X)$ is said to be a function space on X if E satisfies the following conditions:

(i) E is a linear subspace of $C(X)$;

(ii) $1 \in E$;

(iii) E separates the points of X; that is, for each pair of distinct points x_1, x_2 in X, there exists f in E such that $f(x_1) \neq f(x_2)$.

Then, a function space E on X, when equipped with the sup-norm becomes a normed real linear space.

Denote by E^* the space of all continuous real linear functionals on E. For λ in E^*, we define its functional norm by

$$\|\lambda\| = \sup\{|\lambda(f)|: \|f\|_\infty \leqq 1, f \in E\} .$$

Let $\mathfrak{M}$ be a set $\{\lambda \in E^*: \|\lambda\| = 1 = \lambda(1)\}$. Then, $\mathfrak{M}$ is a convex set.

For f in E and λ in $\mathfrak{M}$, let $\hat{f}(\lambda) = \lambda(f)$, and let $\hat{E} = \{\hat{f}: f \in E\}$. Then, $\hat{E}$ is a linear space of functions on $\mathfrak{M}$, and the map $f \to \hat{f}$, which we shall call the generalized Gelfand map, is linear from E to $\hat{E}$.

With the weak topology defined by the family $\hat{E}$, $\mathfrak{M}$ becomes a topological space, and in fact $\mathfrak{M}$ is a compact Hausdorff space.

Let h be a real-valued function on a convex set K. The function h is said to be affine if $h(a\lambda + (1-a)\mu) = ah(\lambda) + (1-a)h(\mu)$ for each λ, μ in K and $0 \leqq a \leqq 1$.

Denote by A_K the set of all continuous affine functions on K. In this paper, we give a proof of the following result.

THEOREM. *$\hat{E} = A_{\mathfrak{M}}$ if E is a closed function space on X.*

It is known that when E is a closed function space with some additional conditions, the above theorem holds (See [2]). But the general theorem above holds, and we give its proof in the following sections.

2. Preliminaries.

Let E be a function space on X.

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For each fixed x in X, let $\varphi(x)f=f(x)$ for all f in E. It is obvious that $\varphi(x)\in\mathfrak{M}$. The map φ: $X\rightarrow\mathfrak{M}$ is one-to-one, since E separates the points of X, and continuous by the definition of the topology on $\mathfrak{M}$. Thus φ is a homeomorphism from X onto a closed subset of $\mathfrak{M}$.

Each $\hat{f}$ in $\hat{E}$ assumes its maximum modulus on the image of X by φ, since we have the following maximum principle: $\|\hat{f}\|_\infty=\|f\|_\infty$ for each f in E. The generalized Gelfand map $f\rightarrow\hat{f}$ is a linear isomorphism and an isometry from E onto $\hat{E}$. Consequently, $\hat{E}$ is a function space on $\mathfrak{M}$, and furthermore, if E is closed, then $\hat{E}$ is also closed.

3. Proof of the Theorem.

For our proof of the theorem, an essential key is the following lemma.

LEMMA. *Let L be a locally convex Hausdorff topological linear space, and K a compact convex subset of L.*

Then, A_K is a closed linear subspace of $C(K)$, and putting

$$L^*|_K+\mathbf{R}=M\,,$$

M is dense in A_K. (Here, if λ is in L^*, then the restriction of λ to K is denoted by $\lambda\,|\,K$, and $L^*|_K=\{\lambda\,|\,K\colon\lambda\in L^*\}$.)

For a proof of the above lemma, see [4].

Let E be a closed function space on X. On E^*, we define the weak*-topology. Then the topology on $\mathfrak{M}$ defined above is the relative topology on $\mathfrak{M}$ induced by the weak*-topology of E^*.

With weak*-topology on E^*, E^* is itself a locally convex Hausdorff topological space.

Denote by $(E^*)^{W-*}$ the space of all weak*-continuous linear functionals on E^*.

Then, by the above lemma, $(E^*)^{W-*}|_{\mathfrak{M}}+\mathbf{R}$ is dense in $A_{\mathfrak{M}}$. On the other hand, if F is any weak*-continuous linear functional of E^*, there exists f in E such that $F(\mu)=\mu(f)$ for all μ in E^*, and therefore, $F|_{\mathfrak{M}}=\hat{f}$.

It follows that $(E^*)^{W-*}|_{\mathfrak{M}}+\mathbf{R}\subset\hat{E}$, since $\hat{E}$ contains the constant functions. But it is clear that $\hat{E}\subset A_{\mathfrak{M}}$. Hence, we can conclude readily that $\hat{E}=A_{\mathfrak{M}}$.

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